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RESTATEMENT OF FIRST-ORDER SHEAR-DEFORMATION THEORY FOR LAMINATED PLATES

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Abstract—A restatement of the first-order shear-deformation theory of plates is offered and verified numerically by exact 3-D elasticity results. Based on a more appropriate physical assumption, the restated theory innovatively interprets its variables and applies elasticity equations in a more pertinent manner. It assumes physically that only in some average sense does a straight line originally normal to the midplane remain straight and rotate relative to the normal of the midplane after deformation. Hence the in-plane displacement is still approximated, in an average sense, as linear and the transverse deflection as constant through the plate thickness. The associated nominal-uniform transverse shear strain directly derived from these displacement field assumptions is identified as the weighted-average transverse shear strain through the plate thickness with the corresponding transverse shear stress as the weighting function, while the actual transverse shear strain is permitted to vary through the thickness and satisfies the constitutive law with its stress counterpart. Likewise, the average rotation of the line is identified as its weighted-average value, instead of the one evaluated from a linear regression of the inplane displacement with the least-square method. Examination of bending energy and transverse shear energy supports this interpretation. In addition, an effective transverse shear stiffness parameter is identified and proven appropriate. This restated first-order, shear-deformation theory yields accurate local as well as global response predictions without employing a shear-correction factor. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Many 2-D shear-deformable laminated plate theories have been proposed to account for appropriate distributions of transverse shear strains through the plate thickness. These 2-D theories can be classified as equivalent single-layer theories (first-order or higher) and layerwise theories. Traditional first-order shear-deformation theory (FSDT) was proposed by Reissner (1945) for isotropic materials and is based on the assumptions that the inplane stresses are distributed linearly over the thickness of the plate. Transverse shear stresses exhibit parabolic distributions. These stress field assumptions are consistent with the displacement field assumptions which include linearly distributed inplane displacements and constant transverse deflection over the plate thickness, which is later explicitly restated by Reissner (1950). Such displacement field assumptions are also assumed by Bolle (1947) and Mindlin (1951). The displacement-based FSDT is more widely used today as the *de facto* version of FSDT, even though the displacement-based FSDT leads to uniform transverse shear strains through the plate thickness and requires a shear correction factor to accommodate parabolic transverse shear stresses. It is widely regarded that FSDT is inadequate for local parameter prediction when applied to laminated plates in that the transverse or interlaminar stresses recovered from the constitutive relations are not continuous through the thickness of the laminate. Similarly, higher-order theories [e.g. Whitney and Sun (1973), Nelson and Lorch (1974), Lo *et al.* (1977), Bert (1984), Reddy (1984), Tessler (1993)] which employ higher-order through-the-thickness expansions for the inplane displacement fields generally violate continuity of transverse shear stresses at dissimilar interfaces of laminated plates if transverse shear stresses are recovered from the constitutive relations. Layerwise theories utilize piecewise interpolation functions (first-order or higher order) through the plate thickness for displacement fields and permit transverse shear strain discontinuity at layer interfaces in an attempt to satisfy both continuity of interlaminar transverse shear stresses and the constitutive equations simultaneously. Certain layerwise theories [e.g. Srinivas (1973), Di Sciuva (1986), Reddy *et al.* (1989)] offer better approximations of transverse shear strains than equivalent single-layer theories do. However, most layerwise theories also use a large number of independent unknowns which makes these theories less attractive in engineering applications. One exception is the zig-zag theory developed by Di Sciuva (1986).

Recently, Qi and Knight (1996) presented a refined first-order, shear-deformation theory for laminated plates which allows the transverse shear strains to vary through the thickness while still assuming a linear inplane displacement expansion and a constant transverse deflection through the thickness, as traditional FSDT does. In contrast to Reissner–Mindlin's FSDT where the constitutive relation between transverse shear stress and shear strain in the pointwise form is not satisfied, Qi–Knight's theory allows the transverse shear strain to vary through the thickness and enforces transverse shear constitutive law in a pointwise manner. The constant or uniform transverse shear strain derived directly from the displacement field assumption is interpreted as the weighted-average value of transverse shear strain through the thickness and the weighting function is the corresponding transverse shear stress, which is based on equivalent shear strain energy. This refined theory accounts for piecewise quadratic distribution of transverse shear strain for laminated plates and satisfies the continuity requirement of the transverse shear stress at layer interfaces. A shear-correction factor, upon which the accuracy of traditional FSDT strongly depends, is thus not involved. Without introducing any new unknowns, Qi–Knight's theory yields excellent agreement for both global and local response parameters (deflection, transverse shear strain and stress distributions) when compared with Pagano's (1969) exact elasticity solution for cylindrical bending problem of symmetric cross-ply laminated plates.

This paper further restates the first-order, shear-deformation theory of plates and examines its validity. It is physically assumed that only in some average sense does a straight line originally normal to the midplane remain straight and rotate relative to the normal of the midplane after deformation. Hence the inplane displacement is still approximated, in an average sense, as linear and the transverse deflection as constant through the plate thickness. The nominal–uniform transverse shear strain, which is directly obtained from the assumed displacement expansion and strain–displacement relation, is identified as the weighted-average value of shear strain through the plate thickness with the corresponding transverse shear stress as the weighting function. Likewise, the average rotation is identified as the weighted-average value of rotation, rather than the simple average one which is obtained from the linear regression of inplane displacement with the least-square method. Numerical results of various expressions of bending strain energy and transverse shear strain energy are also compared, which support the restatement of FSDT for laminated plates given herein.

THEORETICAL ANALYSIS

Reissner–Mindlin's FSDT assumes that a line initially normal to the midplane of a plate remains straight but not necessarily normal after bending. That is, the line is permitted to rotate relative to the normal of the midplane, and the relative rotation angle is the *nominal–uniform transverse shear strain.* Since such an assumption succeeds in predicting the global response but falls short in predicting local response, the restated FSDT retains this assumption of Reissner–Mindlin's FSDT only in some average or global sense. Namely, that line rotates during the bending deformation and remains straight only in some average sense, and this relative rotation angle corresponds to some average value of transverse shear strain through the plate thickness. The actual transverse shear strain, which can no longer be directly derived from the displacement assumptions, can be evaluated from the constitutive law and transverse shear stress, while the transverse shear stress is determined from the integration of the equilibrium equation along with the inplane stress.

For the sake of simplicity, a bending problem of a semi-infinite, symmetric, orthotropic laminated plate with the total thickness of $2h$ is considered. The plate configuration is

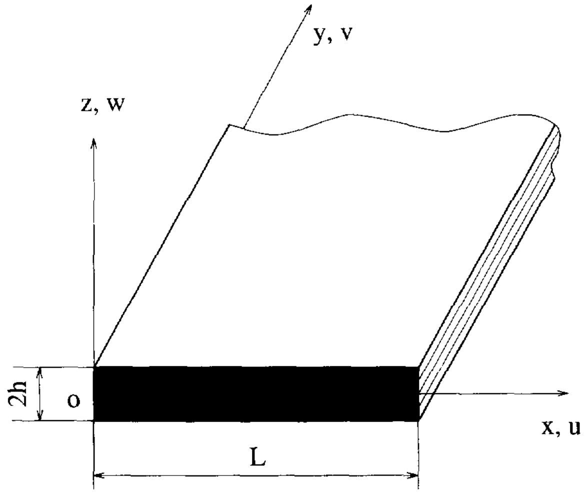


Fig. 1. Plate configuration.

illustrated in Fig. 1. The displacement field, in some average sense, is assumed to take the form

$$
\begin{aligned}
u_x(x,z) &= z\bar{\theta}(x) \\
u_z(x,z) &= w(x)
\end{aligned}
\tag{1}
$$

where u_x, u_z are displacements in the x, z directions, respectively, $\bar{\theta}(x)$ is the average rotation angle of a deformed line relative to its initial undeformed position, and $w(x)$ is the transverse deflection. Both $\bar{\theta}(x)$ and $w(x)$ are independent of the thickness coordinate z.

A nondimensional thickness coordinate ζ is also introduced and given by

$$\zeta = \frac{z}{h} \quad \zeta \in [-1, 1]. \tag{2}$$

Equation (1) gives rise to a linear distribution of inplane normal strain ε_x which is generally adequate to represent the actual one. That is,

$$\varepsilon_x(x,z) = \frac{\partial u_x}{\partial x} = z\bar{\theta}_{,x}. \tag{3}$$

Using a constitutive law, one can obtain the corresponding inplane normal stress,

$$\sigma_x(x,z) = Q_{11}^{(k)}\varepsilon_x = zQ_{11}^{(k)}\bar{\theta}_{,x} \tag{4}$$

where $Q_{11}^{(k)}$ is the reduced stiffness coefficient for the kth layer.

One can express the through-the-thickness transverse shear strain from strain–displacement relation as

$$\gamma_{xz}(x,z) = \frac{\partial u_x(x,z)}{\partial z} + \frac{\partial u_z(x,z)}{\partial x}. \tag{5}$$

Substituting eqn (1) into eqn (5) gives rise to a constant or uniform transverse shear strain through the plate thickness, which is referred to herein as the *nominal–uniform transverse shear strain* $\bar{\gamma}_{xz}$. That is,

$$\bar{\gamma}_{xz}(x) = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = \bar{\theta} + w_{,x}. \tag{6}$$

This nominal–uniform transverse shear strain is interpreted herein as an average value of the actual nonuniform transverse shear strain $\gamma_{xz}(x, z)$ through the plate thickness in some weighted sense which will be identified soon. The transverse shear strain energy per unit length $U_s(x)$ can be expressed in terms of either the nonuniform transverse shear strain $\gamma_{xz}(x, z)$ as

$$U_s(x) = \frac{1}{2}\int_{-h}^{h} \tau_{xz}(x, z)\gamma_{xz}(x, z)\,\mathrm{d}z \tag{7}$$

or in terms of nominal–uniform (or average) transverse shear strain $\bar{\gamma}_{xz}(x)$ which is independent of the thickness coordinate, the transverse shear strain energy $\bar{U}_s(x)$ is given by

$$\bar{U}_s(x) = \frac{1}{2}\bar{\gamma}_{xz}(x)\int_{-h}^{h} \tau_{xz}(x, z)\,\mathrm{d}z. \tag{8}$$

Based on equivalent shear strain energy, $U_s(x) = \bar{U}_s(x)$, one can easily identify the nominal–uniform transverse shear strain $\bar{\gamma}_{xz}(x)$ in FSDT as the weighted-average value of the actual transverse shear strain through the plate thickness, and the weighting function is the corresponding transverse shear stress. That is,

$$\bar{\gamma}_{xz}(x) = \frac{\int_{-h}^{h} \tau_{xz}(x, z)\gamma_{xz}(x, z)\,\mathrm{d}z}{\int_{-h}^{h} \tau_{xz}(x, z)\,\mathrm{d}z}. \tag{9}$$

Then from eqns (6) and (9), the average rotation angle $\bar{\theta}(x)$ is identified as the weighted-average value of the actual rotation angle $\theta(x, z)$ as expressed by

$$\bar{\theta}(x) = \bar{\gamma}_{xz}(x) - \frac{\partial w(x)}{\partial x} = \frac{\int_{-h}^{h} \tau_{xz}(x, z)\left[\gamma_{xz}(x, z) - \frac{\partial w(x)}{\partial x}\right]\mathrm{d}z}{\int_{-h}^{h} \tau_{xz}(x, z)\,\mathrm{d}z}$$

$$= \frac{\int_{-h}^{h} \tau_{xz}(x, z)\theta(x, z)\,\mathrm{d}z}{\int_{-h}^{h} \tau_{xz}(x, z)\,\mathrm{d}z} \tag{10}$$

where the actual rotation $\theta(x, z)$ is defined as

$$\theta(x, z) = \frac{\partial u_x(x, z)}{\partial z} = \gamma_{xz}(x, z) - \frac{\partial u_z(x, z)}{\partial x} = \gamma_{xz}(x, z) - \frac{\partial w(x)}{\partial x} \tag{11}$$

by using eqn (5) and the inextensibility in the thickness direction as expressed in eqn (1).

The actual transverse shear strain $\gamma_{xz}(x, z)$ is related by the constitutive relation to the transverse shear stress $\tau_{xz}(x, z)$, which itself can be determined from the integration of the equilibrium equation along with the inplane normal stress expressed by eqn (4). That is,

$$\tau_{xz}(x,z) = \tau_{xz}(x,h) - \int_h^z \frac{\partial \sigma_x(x,z)}{\partial x}\,\mathrm{d}z = \frac{\mathrm{d}^2\bar{\theta}(x)}{\mathrm{d}x^2}\int_z^h zQ_{11}^{(k)}\,\mathrm{d}z \tag{12}$$

where the traction-free boundary conditions $[\tau_{xz}(x, \pm h) = 0]$ are imposed and no body force is assumed.

It is evident from eqn (12) that the transverse shear stress can be decomposed into two parts—one part is a function of x only and the other of z only, as used by Brebbia and Connor (1973). The part which is a function of z represents the through-the-thickness distribution of transverse shear stress. As in Qi and Knight (1966), the through-the-thickness distribution of transverse shear stress is represented by the normalized function $H_\tau(\zeta)$ and expressed by

$$H_\tau(\zeta) = \frac{4h}{3V(x)}\tau_{xz}(x,\zeta) = \frac{\int_\zeta^1 \eta Q_{11}^{(k)}\,\mathrm{d}\eta}{\frac{3}{4}\int_{-1}^{1}\left(\int_\zeta^1 \eta Q_{11}^{(k)}\,\mathrm{d}\eta\right)\mathrm{d}\zeta} \tag{13}$$

where $V(x)$ is the transverse shear stress resultant.

One can use transverse shear constitutive relation to determine the normalized through-the-thickness distribution function of transverse shear strain $H_\gamma(\zeta)$. That is,

$$H_\gamma(\zeta) = \frac{\dfrac{H_\tau(\zeta)}{c_{55}^{(k)}}}{\dfrac{3}{4}\displaystyle\int_{-1}^{1}\frac{H_\tau(\zeta)}{c_{55}^{(k)}}\,\mathrm{d}\zeta}. \tag{14}$$

For homogeneous plates, it can be shown that

$$H_\tau(\zeta) = H_\gamma(\zeta) = 1-\zeta^2 \tag{15}$$

which is consistent with the constitutive law and compatible with the traction-free boundary conditions. Both $H_\tau(\zeta)$ and $H_\gamma(\zeta)$ are normalized in that

$$\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta = \int_{-1}^{1} H_\gamma(\zeta)\,\mathrm{d}\zeta = \int_{-1}^{1}(1-\zeta^2)\,\mathrm{d}\zeta = \frac{4}{3} \tag{16}$$

which is why a factor 3/4 or 4/3 appears in eqns (13) and (14).

The actual transverse shear strain $\gamma_{xz}(x,\zeta)$ can then be expressed as the product of the nominal–uniform transverse shear strain $\bar{\gamma}_{xz}(x)$, a constant factor f_γ, and the normalized distribution shape function $H_\tau(\zeta)$ based on equivalent transverse shear strain energy. That is,

$$\gamma_{xz}(x,\zeta) = \frac{\int_{-1}^{1} H_\tau(\zeta)\,\mathrm{d}\zeta}{\int_{-1}^{1} H_\tau(\zeta)H_\gamma(\zeta)\,\mathrm{d}\zeta}\bar{\gamma}_{xz}(x)H_\gamma(\zeta) = f_\gamma\bar{\gamma}_{xz}(x)H_\gamma(\zeta). \tag{17}$$

The transverse shear strain energy per unit length $U_s(x)$ can be restated in terms of the nominal–uniform transverse shear strain and a corresponding effective transverse shear stiffness parameter F such that

$$U_s(x) = \frac{1}{2}\int_{-1}^{1} \tau_{xz}(x,\zeta)\gamma_{xz}(x,\zeta)h\,\mathrm{d}\zeta = \frac{1}{2}F\bar{\gamma}_{xz}^2(x) \tag{18}$$

where

$$F = hf_\gamma^2 \int_{-1}^{1} c_{55}^{(k)} H_\gamma^2(\zeta)\,\mathrm{d}\zeta. \tag{19}$$

Using these definitions, the effective transverse shear stiffness parameter F and transverse shear strain energy U_s have forms analogous to the bending stiffness parameter D and bending strain energy U_b, respectively.

For a semi-infinite laminated plate, $H_\tau(\zeta)$, $H_\gamma(\zeta)$, and F are available *a priori*. Given certain types of loadings and boundary conditions, one can easily obtain displacement variables $w(x)$ and $\bar{\theta}(x)$ using the Rayleigh–Ritz method with the transverse shear strain energy per unit length expressed by eqn (18). The actual transverse shear strain $\gamma_{xz}(x,z)$ can be directly evaluated from eqn (17). Since the actual transverse shear strain distribution is properly accounted for through the plate thickness and the traction-free boundary conditions are automatically satisfied, the use of a shear correction factor is no longer required in this restated FSDT.

NUMERICAL RESULTS

The cylindrical bending problems of symmetric orthotropic laminated plates used by Qi and Knight (1996) are solved with the aforementioned restated theory. Supplemental results are provided.

The composite layers are assumed to have the following stiffness properties, which simulate a high-modulus graphite/epoxy composite,

$$E_L = 25\times10^6\ \text{psi},\quad E_T = 10^6\ \text{psi},\quad G_{LT} = 0.5\times10^6\ \text{psi}$$
$$G_{TT} = 0.2\times10^6\ \text{psi},\quad \nu_{LT} = \nu_{TT} = 0.25 \tag{20}$$

where L and T denote the longitudinal and transverse ply material directions, respectively.

Three plates are considered: a single [0] layer plate; a 4-layer laminate $[90/0]_s$ plate; and a 16-layer laminate $[90_3/0_3/90/0]_s$ plate. In all cases, each layer is assumed to have the same thickness—0.005 in, while the span of each plate, denoted by L, varies in order to maintain ratios of span-to-thickness $[R = L/(2h)]$ equal to 4, 10, 20 and 50, for different plate geometry configurations.

For all cylindrical bending problems, the lateral load is assumed to be equally distributed over both the top and bottom surfaces of the plates. That is,

$$p(x,\pm h) = \pm\frac{1}{2}p_0\sin\frac{\pi x}{L} \tag{21}$$

where $p_0 = 100$ psi in all cases. The semi-infinite plates are simply supported at $x = 0, L$. The variables $w(x)$ and $\bar{\theta}(x)$ are readily obtained using the restated FSDT. For the same bending problems, exact solutions are also available from Pagano (1969) and serve as benchmark solutions. All the following results are evaluated at $x = L/4$.

Results are shown in Tables 1–3, comparing average rotation angles $\bar{\theta}$, bending strain energies U_b, average transverse shear strains $\bar{\gamma}_{xz}$, and transverse shear strain energies U_s for each plate. These results are presented in terms of relative error measures defined as

Table 1. Comparisons of (a) rotation angle and bending strain energy and (b) transverse shear strain and strain energy for single [0] layer plates

(a)

$L/(2h)$	$\delta_{\bar{\theta}}^{(b)}$	$\delta_{\bar{\theta}}^{(c)}$	$\delta_{U_b}^{(b)}$	$\delta_{U_b}^{(c)}$	$\delta_{U_b}^{(d)}$
4	−5.52	−4.82	−5.06	−3.64	6.36
10	0.08	0.24	−0.08	0.25	−0.24
20	0.08	0.12	−0.03	0.06	−0.18
50	0.01	0.02	−0.07	−0.06	−0.10

(b)

$L/(2h)$	$\delta_{\bar{\gamma}_{xz}}^{(b)}$	$\delta_{\bar{\gamma}_{xz}}^{(c)}$	$\delta_{U_s}^{(b)}$	$\delta_{U_s}^{(c)}$	$\delta_{U_s}^{(d)}$
4	5.10	0.55	−4.80	−1.47	5.16
10	1.03	−0.46	−0.98	−1.96	1.07
20	0.23	−0.76	−0.18	−2.12	0.49
50	−0.14	−0.85	−0.05	−2.17	0.03

Table 2. Comparisons of (a) rotation angle and bending strain energy and (b) transverse shear strain and strain energy for $[90/0]_s$ laminated plates

(a)

$L/(2h)$	$\delta_{\bar{\theta}}^{(b)}$	$\delta_{\bar{\theta}}^{(c)}$	$\delta_{U_b}^{(b)}$	$\delta_{U_b}^{(c)}$	$\delta_{U_b}^{(d)}$
4	−0.25	17.64	−1.19	37.42	−0.70
10	0.33	3.54	−0.33	6.16	−0.98
20	0.10	0.91	−0.45	1.18	−0.64
50	0.02	0.15	−0.49	−0.23	−0.52

(b)

$L/(2h)$	$\delta_{\bar{\gamma}_{xz}}^{(b)}$	$\delta_{\bar{\gamma}_{xz}}^{(c)}$	$\delta_{U_s}^{(b)}$	$\delta_{U_s}^{(c)}$	$\delta_{U_s}^{(d)}$
4	5.71	−23.10	−5.15	−43.62	5.99
10	0.74	−26.49	−0.48	−47.32	0.99
20	−0.06	−27.04	0.32	−47.90	0.19
50	−0.29	−27.20	0.54	−48.06	0.04

Table 3. Comparisons of (a) rotation angle and bending strain energy and (b) transverse shear strain and strain energy for $[90_3/0_3/90/0]_s$ laminated plates

(a)

$L/(2h)$	$\delta_{\bar{\theta}}^{(b)}$	$\delta_{\bar{\theta}}^{(c)}$	$\delta_{U_b}^{(b)}$	$\delta_{U_b}^{(c)}$	$\delta_{U_b}^{(d)}$
4	−5.33	26.95	−5.47	69.99	−9.11
10	0.15	6.21	−0.07	12.40	−0.36
20	0.08	1.62	−0.05	−1.91	−0.22
50	0.02	0.26	−0.09	−0.32	0.00

(b)

$L/(2h)$	$\delta_{\bar{\gamma}_{xz}}^{(b)}$	$\delta_{\bar{\gamma}_{xz}}^{(c)}$	$\delta_{U_s}^{(b)}$	$\delta_{U_s}^{(c)}$	$\delta_{U_s}^{(d)}$
4	10.11	−18.95	−9.11	−39.70	10.19
10	1.72	−23.02	−1.64	−42.20	1.78
20	0.40	−23.70	−0.35	−42.61	0.45
50	0.02	−23.89	0.03	−42.73	0.09

$$\delta_{\Psi}^{(i)} = \left(1 - \frac{\Psi^{(i)}}{\Psi^{(a)}}\right) \times 100$$

$$\Psi = \bar{\theta}, U_b, \bar{\gamma}_{xz}, U_s; \quad i = b, c, d \tag{22}$$

where Ψ stands for one of the four quantities being considered (i.e. $\bar{\theta}$, $U_b, \bar{\gamma}_{xz}$ or U_s), and the superscript (i) denotes the different approach used to evaluated these quantities. Variables with a superscript (a) denote those obtained from the analytical elasticity solutions. Variables with a superscript (b) denote those obtained from this present restated FSDT. Variables with a superscript (c) denote those obtained using least-square method to evaluate average rotation angle. Strain energies with a superscript (d) denote those obtained using the weighted-average values of rotations or transverse shear strains. Further details are given in the Appendix.

Table 1 gives the comparison for a single-layer orthotropic homogeneous plate. Tables 2 and 3 present results for two laminated plates. When the plates are at most moderately thick ($R \geqslant 10$), the following conclusions can be made from results in these tables:

1. The nominal–uniform transverse shear strain $\bar{\gamma}_{xz}^{(b)}$ (or the average relative rotation angle) determined from the restated FSDT for the three laminates considered, is very close to the weighted-average transverse shear strain evaluated from the exact results $\bar{\gamma}_{xz}^{(a)}$ with the corresponding transverse shear stress as the weighting function. Their relative error is usually within 2%.
2. The transverse shear strain energy obtained from either the nominal–uniform transverse shear strain $\bar{\gamma}_{xz}^{(b)}$ or the weighted-average shear strain $\bar{\gamma}_{xz}^{(a)}$, namely, either $U_s^{(b)}$ or $U_s^{(d)}$, is very close to the actual value of transverse shear strain energy $U_s^{(a)}$. Their relative error does not exceed 2%.
3. The definition of effective transverse shear stiffness F, as in eqn (19), is appropriate. Along with the nominal–uniform transverse shear strain, the effective transverse shear stiffness is adequate to represent the transverse shear strain energy through the plate thickness.
4. A better approximation of bending strain energy is achieved if the average rotation angle $\bar{\theta}$ with respect to its original undeformed position is interpreted as the transverse-shear-stress-weighted-average one [$\bar{\theta}^{(a)}$, related to $\bar{\gamma}_{xz}^{(a)}$ by eqn (A-3)] than as the linear-regression one [$\bar{\theta}^{(c)}$, corresponding to $\bar{\gamma}_{xz}^{(c)}$]. Such interpretation becomes much more significant when transverse shear strain and transverse shear strain energy are concerned. As indicated in Tables 1–3, the use of $\bar{\gamma}_{xz}^{(a)}$ leads to an accurate approximation of transverse shear strain energy while the use of $\bar{\gamma}_{xz}^{(c)}$ results in a poor one.

In order to further illustrate the validity of the restated FSDT, the transverse shear strain distributions through the laminate thickness are plotted in Figs 2 and 3. The results obtained using the present theory are comparable to those obtained using the exact elasticity solution of Pagano (1969). Also included in these figures is the weighted-average value, $\bar{\gamma}_{xz}^{(a)}$, which is defined in the restated FSDT as, and is actually equal to, the nominal–uniform transverse shear strain, $\bar{\gamma}_{xz}^{(b)}$. The transverse shear strain corresponding to the linear regression of inplane displacement, $\bar{\gamma}_{xz}^{(c)}$, is also plotted for comparison.

In Figs 2 and 3, $W-A$ stands for weighted-average transverse shear strain $\bar{\gamma}_{xz}^{(a)}$ and $L-R$ represents the transverse shear strain from a linear regression analysis $\bar{\gamma}_{xz}^{(c)}$. Results in both figures demonstrate that the present restated FSDT (denoted by *rFSDT*) yields an excellent approximation of transverse shear strain for moderately thick plates ($R = 10$) when compared to the exact results (denoted by *Exact*). When the plates become thinner, the difference, if any, between transverse shear strain predicted using restated FSDT and the exact one can hardly be detected if plotted on such a figure.

Results for the case of thick plates ($R = 4$) are included in these numerical results and figures only for the sake of comparison. To the best knowledge of the authors, first-order

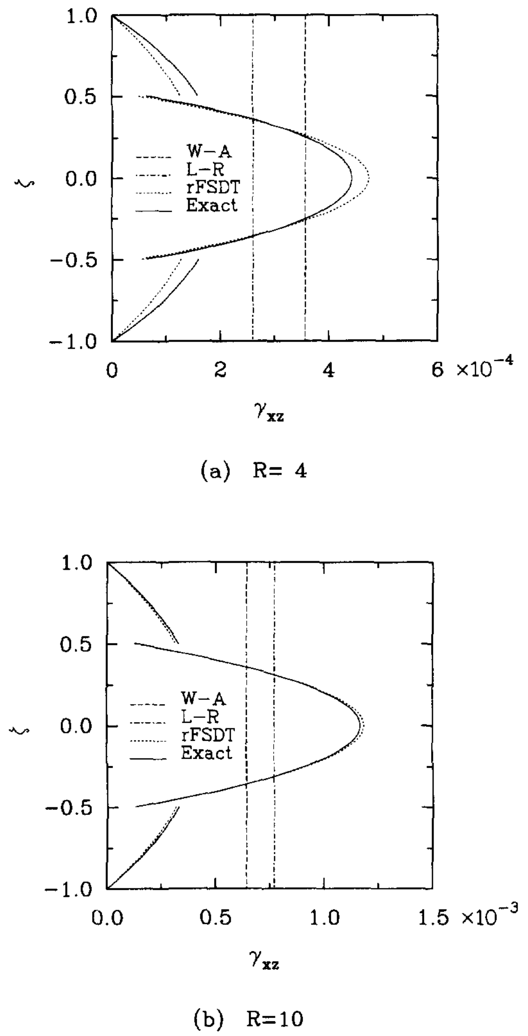


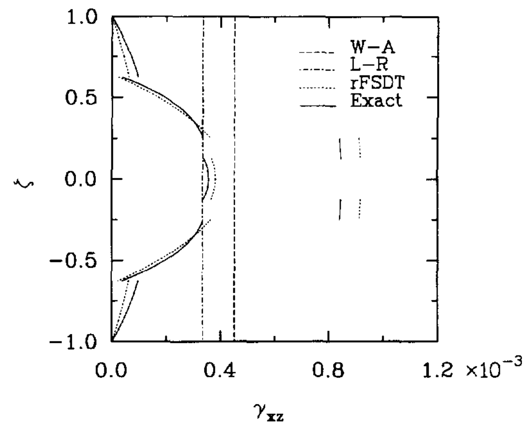
Fig. 2. Transverse shear strain distribution comparison for $[90/0]_s$ laminate.

theory is inadequate for thick laminated plate ($R < 10$) where the inplane displacement exhibits a pronounced nonlinear distribution through the plate thickness.

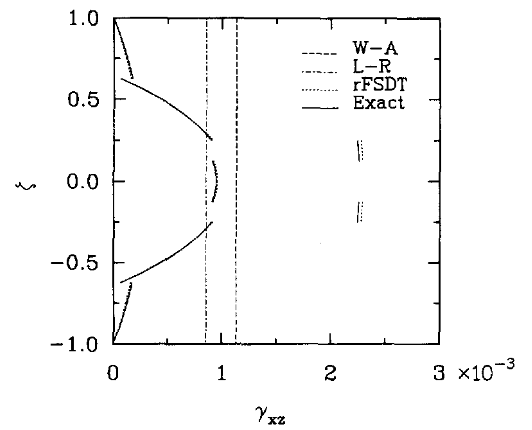
Since the present theory successfully approximates both bending strain energy and transverse shear strain energy, it is beyond doubt that the restated FSDT offers excellent global as well as local response predictions for moderately thick plates ($R \geqslant 10$) wherein the transverse extensional strain energy can be neglected. The predictions for transverse deflection and transverse shear stress obtained using the restated FSDT are given in Qi and Knight (1996).

SUMMARY

In summary, this restatement of first-order shear-deformation theory of plates is embodied in the following key points. First, the plate is assumed to be inextensible in the thickness direction as in FSDT. Second, the inplane displacement is assumed to be a linear distribution through the plate thickness in an average sense. That is, only in some weighted-average sense does a line that is originally straight and normal to the midplane before deformation remain straight and rotate relative to the normal of the midplane after deformation. Third, the inplane strain can be directly derived from the displacement assumptions and the strain–displacement relation. In this statement, the transverse shear strain directly derived from the displacement assumption corresponds to the average relative rotation angle, or nominal–uniform transverse shear strain $\bar{\gamma}_{xz}(x)$, and is equal to the weighted



(a) R= 4



(b) R=10

Fig. 3. Transverse shear strain distribution comparison for $[90_3/0_3/90/0]_s$ laminate.

average transverse shear strain through the thickness with the transverse shear stress as the weighting function.

$$\bar{\gamma}_{xz}(x) = \frac{\int_{-h}^{h} \tau_{xz}(x, z)\gamma_{xz}(x, z)\,\mathrm{d}z}{\int_{-h}^{h} \tau_{xz}(x, z)\,\mathrm{d}z} = \bar{\theta}(x) + w_{,x}(x). \tag{23}$$

The average rotation angle $\bar{\theta}(x)$ with respect to the original undeformed position also corresponds to the weighted-average value. Fourth, the actual transverse shear strain $\gamma_{xz}(x, z)$ is related to the transverse shear stress $\tau_{xz}(x, z)$ pointwise through the plate thickness by the constitutive law, rather than uniform as in FSDT. Finally, the transverse shear stiffness parameter F is defined which is analogous to the bending stiffness parameter D. A shear correction factor is not needed.

The restated FSDT possesses the advantages of both equivalent single-layer theories and layerwise theories. It takes the simplest displacement expansion among all the shear deformable theories and does not require the use of a shear correction factor. It accounts for a variable distribution of transverse shear strain to which higher-order theories are developed. It satisfies proper continuity requirement of transverse shear stress at layer interfaces which layerwise theories are proposed to achieve. The constitutive law and traction boundary conditions are automatically satisfied in the restated FSDT. Effective transverse shear stiffness is also defined in an analogous manner to bending stiffness. As anticipated, excellent agreement of the restated FSDT with the exact elasticity results has been accomplished in terms of both global prediction (transverse deflection and strain energies) and local prediction (transverse shear strain and shear stress). The extension of applying the restated FSDT from cylindrical bending (or plane-strain bending) problem to a bidirectional bending problem is anticipated, though in a more complex form.

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APPENDIX

The actual bending energy $U_b^{(a)}$ and transverse shear strain $U_s^{(a)}$ per unit length are integrated numerically from the exact results of stresses and strains. That is,

$$U_b^{(a)} = \frac{1}{2}\int_{-h}^{h} \sigma_x^{(a)}(x,z)\varepsilon_x^{(a)}(x,z)\,\mathrm{d}z.$$

$$U_s^{(a)} = \frac{1}{2}\int_{-h}^{h} \tau_{xz}^{(a)}(x,z)\gamma_{xz}^{(a)}(x,z)\,\mathrm{d}z. \tag{A-1}$$

The actual weighted-average transverse shear strain $\bar{\gamma}_{xz}^{(a)}$ is evaluated from the exact elasticity solution,

$$\bar{\gamma}_{xz}^{(a)}(x) = \frac{\displaystyle\int_{-h}^{h} \tau_{xz}^{(a)}(x,z)\gamma_{xz}^{(a)}(x,z)\,\mathrm{d}z}{\displaystyle\int_{-h}^{h} \tau_{xz}^{(a)}(x,z)\,\mathrm{d}z} \tag{A-2}$$

and $\bar{\theta}^{(a)}$ is defined as the actual weighted-average rotation angle. That is,

$$\bar{\theta}^{(a)}(x) = \frac{\displaystyle\int \tau_{xz}^{(a)}\left(\gamma_{xz}^{(a)} - \frac{\partial u_z^{(a)}(x,z)}{\partial x}\right)\mathrm{d}z}{\displaystyle\int \tau_{xz}^{(a)}\,\mathrm{d}z}. \tag{A-3}$$

In eqns (A-1)–(A-3), $\sigma_x^{(a)}, \varepsilon_x^{(a)}, \tau_{xz}^{(a)}, \gamma_{xz}^{(a)}, u_z^{(a)}$ are taken from the exact 3-D elasticity results. Since the transverse deflection shows negligible variation through the thickness, $\bar{\theta}^{(a)}$ can also be rewritten as

$$\bar{\theta}^{(a)}(x) = \bar{\gamma}_{xz}^{(a)}(x) - \frac{\partial u_z^{(a)}(x,0)}{\partial x} \tag{A-4}$$

which is used to generate the data given in the tables.

Variables with a superscript (b) denote those obtained from the present restated FSDT. $\bar{\theta}^{(b)}$ and $w^{(b)}$ are directly obtained from this theory and the other parameters are defined as

$$\begin{aligned} \bar{\gamma}_{xz}^{(b)} &= \bar{\theta}^{(b)} + w_{,x}^{(b)} \\ U_b^{(b)} &= \tfrac{1}{2} D[\bar{\theta}_{,x}^{(b)}]^2 \\ U_s^{(b)} &= \tfrac{1}{2} F[\bar{\gamma}_{xz}^{(b)}]^2 \end{aligned} \tag{A-5}$$

where D is the bending stiffness of the plate and F is the effective transverse shear stiffness defined by eqn (19).

Variables with a superscript (c) denote those parameters corresponding to $\bar{\theta}^{(c)}$ which is the average rotation calculated with least-square method from eqn (1) and the exact inplane displacement results $u_{xz}^{(a)}(x, z)$. That is,

$$\frac{\partial}{\partial \bar{\theta}^{(c)}} \left(\int_{-h}^{h} [u_x^{(a)}(x,z) - z\bar{\theta}^{(c)}]^2 \, dz \right) = 0. \tag{A-6}$$

The other parameters used in the comparisons are defined as

$$\begin{aligned} \bar{\gamma}_{xz}^{(c)} &= \bar{\theta}^{(c)} + \frac{\partial u_z^{(a)}(x,0)}{\partial x} \\ U_b^{(c)} &= \tfrac{1}{2} D[\bar{\theta}_{,x}^{(c)}]^2 \\ U_s^{(c)} &= \tfrac{1}{2} F[\bar{\gamma}_{xz}^{(c)}]^2. \end{aligned} \tag{A-7}$$

The strain energy expressions with a superscript (d) correspond to those obtained from the actual weighted-average rotation angle and actual weighted-average transverse shear strain as given by

$$\begin{aligned} U_b^{(d)} &= \tfrac{1}{2} D[\bar{\theta}_{,x}^{(a)}]^2 \\ U_s^{(d)} &= \tfrac{1}{2} F[\bar{\gamma}_{xz}^{(a)}]^2. \end{aligned} \tag{A-8}$$